

A note on rigidity and triangulability of a derivation

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Abstract: Let A be a \mathbb{Q} -domain, $K = \text{frac}(A)$, $B = A^{[n]}$ and $D \in \text{LND}_A(B)$. Assume $\text{rank } D = \text{rank } D_K = r$, where D_K is the extension of D to $K^{[n]}$. Then we show that

(i) If D_K is rigid, then D is rigid.

(ii) Assume $n = 3$, $r = 2$ and $B = A[X, Y, Z]$ with $DX = 0$. Then D is triangulable over A if and only if D is triangulable over $A[X]$. In case A is a field, this result is due to Daigle.

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1 Introduction

Throughout this paper, k is a field and all rings are \mathbb{Q} -domains. We will begin by setting up some notations from [4]. Let $B = A^{[n]}$ be an A -algebra, i.e. B is A -isomorphic to the polynomial ring in n variables over A . A *coordinate system* of B over A is an ordered n -tuple (X_1, X_2, \dots, X_n) of elements of B such that $A[X_1, X_2, \dots, X_n] = B$.

An A -derivation $D : B \rightarrow B$ is *locally nilpotent* if for each $x \in B$, there exists an integer $s > 0$ such that $D^s(x) = 0$; D is *triangulable* over A if there exists a coordinate system (X_1, \dots, X_n) of B over A such that $D(X_i) \in A[X_1, \dots, X_{i-1}]$ for $1 \leq i \leq n$; *rank* of D is the least integer $r \geq 0$ for which there exists a coordinate system (X_1, \dots, X_n) of B over A satisfying $A[X_1, \dots, X_{n-r}] \subset \ker D$; $\text{LND}_A(B)$ is the set of all locally nilpotent A -derivations of B .

Let $\Gamma(B)$ be the set of coordinate systems of B over A . Given $D \in \text{LND}_A(B)$ of rank r , let $\Gamma_D(B)$ be the set of $(X_1, \dots, X_n) \in \Gamma(B)$ satisfying $A[X_1, \dots, X_{n-r}] \subset \ker D$; D is *rigid* if $A[X_1, \dots, X_{n-r}] = A[X'_1, \dots, X'_{n-r}]$ holds whenever (X_1, \dots, X_n) and (X'_1, \dots, X'_n) belong to $\Gamma_D(B)$.

For an example, if $D \in \text{LND}_A(B)$ has rank 1, then D is rigid. In this case $\ker(D) = A[X_1, \dots, X_{n-1}]$ for some coordinate system (X_1, \dots, X_n) and $D = f\partial_{X_n}$ for some $f \in \ker(D)$. If $\text{rank } D = n$, then D is obviously rigid, as no variable is in $\ker(D)$. If $\text{rank } D \neq 1, n$, then $\ker(D)$ is not generated by $n - 1$ elements of a coordinate system and is generally difficult to see whether D is rigid. For an example of a non-rigid triangular derivation on $k^{[4]}$, see section 3. We remark that there is also a notion of a ring to be rigid. We say that a ring A is rigid if $\text{LND}(A) = \{0\}$, i.e. there is no non-zero locally nilpotent derivation on A . Clearly polynomial rings $k^{[n]}$ are non-rigid rings for $n \geq 1$.

We will state the following result of Daigle ([4], Theorem 2.5) which is used later.

Theorem 1.1 *All locally nilpotent derivations of $k^{[3]}$ are rigid.*

Our first result extends this as follows:

Theorem 1.2 *Let A be a ring, $B = A^{[n]}$, $K = \text{frac}(A)$ and $D \in \text{LND}_A(B)$. Assume that $\text{rank } D = \text{rank } D_K$, where D_K is the extension of D to $K^{[n]}$. If D_K is rigid, then D is rigid.*

In ([4], Corollary 3.4), Daigle obtained the following triangulability criteria: Let D be an irreducible, locally nilpotent derivation of $R = k^{[3]}$ of rank at most 2. Let $(X, Y, Z) \in \Gamma(R)$ be such that $DX = 0$. Then D is triangulable over k if and only if D is triangulable over $k[X]$. Our second result extends this result as follows:

Theorem 1.3 *Let A be a ring, $B = A^{[3]}$, $K = \text{frac}(A)$ and $D \in \text{LND}_A(B)$. Let $(X, Y, Z) \in \Gamma(B)$ be such that $DX = 0$. Assume that $\text{rank } D = \text{rank } D_K = 2$. Then D is triangulable over A if and only if D is triangulable over $A[X]$.*

2 Preliminaries

Recall that a ring is called a *HCF*-ring if intersection of two principal ideal is again a principal ideal. We state some results for later use.

Lemma 2.1 (Daigle [4], 1.2) *Let D be a k -derivation of $R = k^{[n]}$ of rank 1 and let $(X_1, X_2, \dots, X_n) \in \Gamma(R)$ be such that $k[X_1, X_2, \dots, X_{n-1}] \subset \ker D$. Then*

- (i) $\ker D = k[X_1, X_2, \dots, X_{n-1}]$;
- (ii) D is locally nilpotent if and only if $D(X_n) \in \ker D$.

Proposition 2.2 (Abhyankar, Eakin and Heinzer [1], Proposition 4.8) *Let R be a HCF-ring, A a ring of transcendence degree one over R and $R \subset A \subset R^{[n]}$ for some $n \geq 1$. If A is a factorially closed subring of $R^{[n]}$, then $A = R^{[1]}$.*

Lemma 2.3 (Abhyankar, Eakin and Heinzer [1], 1.7) *Suppose $A^{[n]} = R = B^{[n]}$. If $b \in B$ is such that $bR \cap A \neq 0$, then $b \in A$.*

Theorem 2.4 ([6], Theorem 4.11) *Let R be a HCF-ring and $0 \neq D \in \text{LND}_R(R[X, Y])$. Then there exists $P \in R[X, Y]$ such that $\ker D = R[P]$.*

Theorem 2.5 (Bhatwadekar and Dutta [3]) *Let A be a ring and $B = A^{[2]}$. Then $b \in B$ is a variable of B over A if and only if for every prime ideal \mathfrak{p} of A , $\bar{b} \in \bar{B} := B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is a variable of \bar{B} over $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.*

3 Rigidity

Theorem 3.1 *Let A be a ring, $B = A^{[n]}$, $K = \text{frac}(A)$ and $D \in \text{LND}_A(B)$. Assume that $\text{rank } D = \text{rank } D_K$, where D_K is the extension of D to $K^{[n]}$. If D_K is rigid, then D is rigid.*

Proof Assume $\text{rank } D = \text{rank } D_K = r$ and D_K is rigid. We need to show that D is rigid, i.e. if (x_1, \dots, x_n) and (y_1, \dots, y_n) are two coordinate systems of B satisfying $A[x_1, \dots, x_{n-r}] \subset \ker D$ and $A[y_1, \dots, y_{n-r}] \subset \ker (D)$, then we have to show that $A[x_1, \dots, x_{n-r}] = A[y_1, \dots, y_{n-r}]$. By symmetry, it is enough to show that $A[x_1, \dots, x_{n-r}] \subset A[y_1, \dots, y_{n-r}]$.

Since D_K is rigid and $\text{rank } D_K = r$, we get $K[x_1, \dots, x_{n-r}] = K[y_1, \dots, y_{n-r}]$. If $f \in A[x_1, \dots, x_{n-r}]$, then $f \in K[y_1, \dots, y_{n-r}]$. We can choose $a \in A$ such that $af \in A[y_1, \dots, y_{n-r}]$ and hence $fB \cap A[y_1, \dots, y_{n-r}] \neq 0$. Applying (2.3) to $A[x_1, \dots, x_{n-r}]^{[r]} = B = A[y_1, \dots, y_{n-r}]^{[r]}$, we get $f \in A[y_1, y_2, \dots, y_{n-r}]$. Therefore $A[x_1, \dots, x_{n-r}] \subset A[y_1, \dots, y_{n-r}]$. This completes the proof. \square

The following result is immediate from (3.1) and (1.1).

Corollary 3.2 *Let A be a ring, $B = A^{[3]}$, $D \in \text{LND}_A(B)$. If $\text{rank } D = \text{rank } D_K$, then D is rigid.*

Remark 3.3 (1) If $D \in \text{LND}_A(B)$, then $\text{rank } D$ and $\text{rank } D_K$ need not be same. For an example, consider $A = \mathbb{Q}[X]$ and $B = A[T, Y, Z]$. Define $D \in \text{LND}_A(B)$ as $DT = 0$, $D(Y) = X$ and $D(Z) = Y$. Then $\text{rank } D = 2$ and $\text{rank } D_K = 1$. Further, $(T' = T - Y^2 + 2XZ, Y, Z) \in \Gamma_D(B)$ and $A[T] \neq A[T']$. Therefore, D is not rigid, whereas D_K is rigid, by (1.1).

Above example gives a $D \in \text{LND}(k^{[4]})$ which is not rigid. Hence Daigle's result (1.1) is best possible. Note that D is a triangular derivation and by [2], $\ker(D)$ is a finitely generated k -algebra.

(2) The condition in (3.1) is sufficient but not necessary, i.e. $D \in \text{LND}_A(B)$ may be rigid even if $\text{rank } D \neq \text{rank } D_K$. For an example consider $A = \mathbb{Q}[X]$ and $B = A[Y, Z]$. Define $D \in \text{LND}_A(B)$ as $D(Y) = X$ and $D(Z) = Y$. Then $\text{rank } D = 2$ and hence D is rigid. Further, $\text{rank } D_K = 1$ and D_K is also rigid, by (1.1).

(3) It will be interesting to know if $D \in \text{LND}(k^{[n]})$ being rigid implies that $\ker(D)$ is a finitely generated k -algebra. The following example could provide an answer.

Let $D = X^3\partial_S + S\partial_T + T\partial_U + X^2\partial_V \in \text{LND}(B)$, where $B = k^{[5]} = k[X, S, T, U, V]$. Daigle and Freudenberg [5] have shown that $\ker(D)$ is not a finitely generated k -algebra. We do not know if D is rigid. We will show that $\text{rank } D = 3$.

Clearly $X, S - XV \in \ker(D)$ is part of a coordinate system. Hence $\text{rank } D \leq 3$. If $\text{rank } D = 1$, then there exists a coordinate system (X_1, \dots, X_4, Y) of B over k such that $X_1, \dots, X_4 \in \ker(D)$. Hence $D = f\partial_Y$ for some $f \in k[X_1, \dots, X_4]$ and $\ker(D) = k[X_1, \dots, X_4]$ is a finitely generated k -algebra, a contradiction. If $\text{rank } D = 2$, then there exists a coordinate system (X_1, X_2, X_3, Y, Z) of B over k such that $X_1, X_2, X_3 \in \ker(D)$. If we write $A = k[X_1, X_2, X_3]$, then $D \in \text{LND}_A(A[Y, Z])$. Since A is UFD, by ([6], Theorem 4.11), $\ker(D) = A^{[1]}$, hence $\ker(D)$ is a finitely generated k -algebra, a contradiction. Therefore, $\text{rank } D$ is 3.

4 Triangulability

We begin with the following result which is of independent interest.

Lemma 4.1 *Let A be a UFD, $K = \text{frac}(A)$, $B = A^{[n]}$ and $D \in \text{LND}_A(B)$. Let D_K be the extension of D on $K^{[n]}$. If D is irreducible, then D_K is irreducible.*

Proof We prove that if D_K is reducible, then so is D . Let $D_K(K^{[n]}) \subset fK^{[n]}$ for some $f \in B$. If $B = A[x_1, \dots, x_n]$, then we can write $Dx_i = fg_i/c_i$ for some $g_i \in B$ and $c_i \in A$ with $\gcd_B(g_i, c_i) = 1$. Since $Dx_i \in B$, we get c_i divides f in B . If c is lcm of c_i 's, then c divides f . If we take $f' = f/c \in B$, then $Dx_i \in f'B$ and hence D is reducible. \square

Proposition 4.2 *Let A be a ring, $B = A^{[3]}$, and $D \in \text{LND}_A(B)$ be of rank one. Let $(X, Y, Z) \in \Gamma(B)$ be such that $DX = 0$. Assume that either A is a UFD or D is irreducible. Then D is triangulable over $A[X]$.*

Proof As $\text{rank } D = 1$, there exists $(X', Y', Z') \in \Gamma(B)$ such that $DX' = DY' = 0$. By (2.1), $\ker D = A[X', Y']$ and $DZ' \in \ker D$.

(i) Assume A is a UFD. Since $A[X] \subset A[X', Y'] \subset A[X]^{[2]}$ and $A[X', Y']$ is factorially closed in $A[X]^{[2]}$; by (2.2), $A[X', Y'] = A[X][P]$ for some $P \in B$. Hence $B = A[X, P, Z']$ and $DZ' \in A[X, P]$. Thus D is triangulable over $A[X]$.

(ii) Assume D is irreducible. Then DZ' must be a unit. To show that X is a variable of $A[X', Y']$ over A . By (2.5), it is enough to prove that for every prime ideal \mathfrak{p} of A , if $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ then \overline{X} is a variable of $\kappa(\mathfrak{p})[X', Y']$ over $\kappa(\mathfrak{p})$. Extend D on $A_{\mathfrak{p}}[X, Y, Z]$ and let \overline{D} be D modulo $\mathfrak{p}A_{\mathfrak{p}}$. Then $\ker \overline{D} = \kappa(\mathfrak{p})[X', Y']$. By (2.2), $\ker \overline{D} = \kappa(\mathfrak{p})[X]^{[1]}$. Therefore X is a variable of $A[X', Y']$, i.e. $A[X', Y'] = A[X, P]$ for some $P \in B$. Hence $B = A[X, P, Z']$. Thus D is triangulable over $A[X]$. \square

Proposition 4.3 *Let A be a ring, $K = \text{frac}(A)$, $B = A^{[3]}$ and $D \in \text{LND}_A(B)$. Let $(X, Y, Z) \in \Gamma(B)$ be such that $DX = 0$. Assume $\text{rank } D = \text{rank } D_K = 2$. Then D is triangulable over A if and only if D is triangulable over $A[X]$.*

Proof We need to show only (\Rightarrow) . Suppose that D is triangulable over A . Then there exists $(X', Y', Z') \in \Gamma(B)$ such that $DX' \in A$, $DY' \in A[X']$ and $DZ' \in A[X', Y']$. If $a = DX' \neq 0$, then $D_K(X'/a) = 1$; which implies that $\text{rank } D_K = 1$, a contradiction. Hence $DX' = 0$.

Since D_K is rigid, by (3.1), D is rigid of rank 2. Therefore $A[X] = A[X']$ and D is triangulable over $A[X]$. \square

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